

# A five-dimensional solitary-wave first order nonlinear PDE integrable by dressing method.

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## Abstract

We derive a five-dimensional nonlinear first order matrix PDE which is a generalization of the completely integrable (2+1)-dimensional  $N$ -wave equation. Similar to the  $\bar{\partial}$ -problem, our algorithm is based on the Fredholm-type linear integral equation with the kernel of special form.

## 1 Introduction

The nonlinear Partial Differential Equations (PDE) integrable by the Inverse Spectral Transform Method (ISTM-integrable PDEs) represent a large class of completely integrable nonlinear equations [1]. The dressing method [2] as a special method for solving the nonlinear PDEs was developed in [3, 4] on the basis of Volterra type integral operators. Later, another version of the dressing method involving the Fredholm type integral operator ( $\bar{\partial}$ -problem) was proposed in [5]. The two families of ISTM-integrable PDEs are well studied. These are the (1+1)- and (2+1)-soliton PDEs (Korteweg-de Vries [1, 6], Kadomtsev-Petviashvili [7], Nonlinear Schrödinger [8] and Devi-Stewartson [9] equations should be mentioned as remarkable examples of such equations) and the Self-dual type equations [10, 11]. Recently the equations associated with commuting vector fields were added to this list [12, 13]. Some modifications of the  $\bar{\partial}$ -method increasing the dimensionality of (partially) integrable PDEs are also known, see [14].

In this paper we consider a new modification of the dressing method based on the Fredholm-type operator and derive a five-dimensional first order nonlinear PDE with large solution space, although we do not show its complete integrability. This PDE has a limit to the classical ISTM-integrable (2+1)-dimensional  $N$ -wave equation.

The structure of the paper is following. In Sec.2, we describe our modification of the dressing algorithm and represent the general steps of the derivation of the nonlinear PDE. The available solution space is discussed in Sec.3. The main results are formulated in Sec.4.

## 2 Dressing algorithm

We consider the  $N \times N$  matrix function  $R(x; \lambda, \mu)$  depending on two spectral parameters  $\lambda$  and  $\mu$  and set of auxiliary parameters  $x = (x_1, x_2, \dots)$ , which will be the independent variables of the nonlinear PDEs. We refer to the inverse of  $R$  as the spectral function  $\Psi$ :

$$\Psi(x; \lambda, \mu) = R^{-1}(x, \lambda, \mu), \quad (1)$$

where

$$\int d\Omega(\nu) R(x, \lambda, \nu) R^{-1}(x; \nu, \mu) = I \delta(\lambda - \mu), \quad (2)$$

where  $\Omega(\nu)$  is some scalar measure on the plane of complex spectral parameter  $\nu$ ,  $I$  is the  $N \times N$  identity operator. The  $x$ -dependence can be introduced by the system of linear PDEs

$$R_{x_k}(x; \lambda, \mu) = r(x; \lambda)A^{(k)}q(x; \mu) + Q^{(k)}(\lambda, \mu), \quad (3)$$

where  $F$  is a constant matrix,  $Q^{(k)}$  are functions of spectral parameters independent on  $x$ . The latter assumption is taken for simplicity and, in principle, can be removed. Notice that if  $Q^{(k)} \equiv 0$ , then we obtain (2+1)-dimensional ISTM-integrable  $N$ -wave equation for the matrix field  $v$ ,

$$v = \int d\Omega(\lambda)d\Omega(\nu)q(x; \lambda)\Psi(x; \lambda, \mu)r(x; \mu). \quad (4)$$

The term  $Q^{(k)}$  in (3) leads us to the higher-dimensional nonlinear PDEs.

The compatibility condition of system (3) yields

$$\left( r(x; \lambda)A^{(k)}q(x; \mu) \right)_{x_n} = \left( r(x; \lambda)A^{(n)}q(x; \mu) \right)_{x_k} \quad (5)$$

which can be splitted into the following two equations:

$$r_{x_n}A^{(k)} = r_{x_k}A^{(n)}, \quad (6)$$

$$A^{(k)}q_{x_n} = A^{(n)}q_{x_k}. \quad (7)$$

## 2.1 Derivation of spectral equation

In this section we derive the spectral equation, i.e. PDE for a functions of spectral parameters  $\Psi$ . For this purpose, we differentiate eq.(1) with respect to  $x_k$  obtaining:

$$\Psi_{x_k}(x; \lambda, \mu) = -\chi(x, \lambda)A^{(k)}\tilde{\chi}(x; \mu) - \int d\nu^2 d\tilde{\nu}^2 \Psi(x; \lambda, \nu)Q^k(\nu, \tilde{\nu})\Psi(x; \tilde{\nu}, \mu), \quad (8)$$

where

$$\chi(x, \lambda) = \int d\Omega(\nu)\Psi(x; \lambda, \nu)r(x, \nu), \quad \tilde{\chi}(x, \mu) = \int d\Omega(\nu)q(x; \nu)\Psi(x; \nu, \mu). \quad (9)$$

For the sake of brevity, we denote  $*$  the integration over the "inside pair" of spectral parameters,

$$f * g \equiv \int d\Omega(\nu)f(\lambda, \nu)g(\nu, \mu). \quad (10)$$

### 2.1.1 Additional constraints for function $R$

Having spectral equation (8) we are not able to construct a complete system of nonlinear PDEs. Therefore we have to impose some additional relations on  $R$  in the following form:

$$Q^{(n)} * R - R * Q^{(n)} = rC^{(n)}q, \quad (11)$$

where  $C^{(n)}$  are some diagonal constant matrices. Applying  $\Psi*$  and  $*\Psi$  to eq.(11) we rewrite it as follows:

$$\Psi * Q^{(n)} - Q^{(n)} * \Psi = \Psi * rC^{(n)}q * \Psi, \quad (12)$$

Compatibility of eqs.(11) yields

$$Q^{(n)} * rC^{(m)} = Q^{(m)} * rC^{(n)}, \quad (13)$$

$$C^{(m)} q * Q^{(n)} = C^{(n)} q * Q^{(m)}, \quad (14)$$

$$Q^{(n)} * Q^{(\tilde{n})} = Q^{(\tilde{n})} * Q^{(n)}. \quad (15)$$

Compatibility of eqs.(11) and (3) yields

$$r_{x_k} C^{(n)} = Q^{(n)} * rA^{(k)}, \quad (16)$$

$$C^{(n)} q_{x_k} = -A^{(k)} q * Q^{(n)}. \quad (17)$$

In virtue of eq.(11), spectral equation (8) reads

$$E^{(n)} := \Psi_{x_n} + \chi A^{(n)} \tilde{\chi} + \Psi^{(2)} * Q^{(n)} - \chi^{(2)} C^{(n)} \tilde{\chi} = 0, \quad (18)$$

$$\Psi^{(2)} = (\Psi F) * \Psi, \quad \chi^{(2)} = \Psi^{(2)} * r, \quad \tilde{\chi} = q * \Psi. \quad (19)$$

System (18) together with constraint (12) represent a spectral system for the nonlinear PDEs derived in the next subsection.

## 2.2 Derivation of nonlinear PDEs

First of all, deriving the nonlinear PDE we note that the term with  $Q^{(n)}$  can be eliminated from eq.(18) owing to condition (13). To use this condition, we apply  $*r$  to eq.(18) and consider the following combination:

$$\begin{aligned} E^{(nm)} &:= E^{(n)} * rC^{(m)} - E^{(m)} * rC^{(n)} = \\ &\chi_{x_n} C^{(m)} - \chi_{x_m} C^{(n)} + \chi A^{(n)} v C^{(m)} - \chi A^{(m)} v C^{(n)} - \chi^{(2)} C^{(n)} v C^{(m)} + \\ &\chi^{(2)} C^{(m)} v C^{(n)} - \Psi * r_{x_n} C^{(m)} + \Psi * r_{x_m} C^{(n)}, \end{aligned} \quad (20)$$

where the field  $v$  is defined in (4). Now, applying  $q*$  to eq.(20) we obtain the nonlinear equation without spectral parameters:

$$\hat{E}^{(nm)} := \hat{E}_0^{(nm)} - \hat{F}^{(nm)}, \quad (21)$$

where

$$\begin{aligned} \hat{E}_0^{(nm)} &= v_{x_n} C^{(m)} - v_{x_m} C^{(n)} + v A^{(n)} v C^{(m)} - v A^{(m)} v C^{(n)} - \\ &v^{(2)} C^{(n)} v C^{(m)} + v^{(2)} C^{(m)} v C^{(n)}, \end{aligned} \quad (22)$$

$$\begin{aligned} F^{(nm)} &= \tilde{\chi} * r_{x_n} C^{(m)} - \tilde{\chi} * r_{x_m} C^{(n)} + q_{x_n} * \chi C^{(m)} - q_{x_m} * \chi C^{(n)} = \\ &\tilde{\chi} * r_{x_1} A^{(n)} C^{(m)} - \tilde{\chi} * r_{x_1} A^{(m)} C^{(n)} + A^{(n)} q_{x_1} * \chi C^{(m)} - A^{(m)} q_{x_1} * \chi C^{(n)}. \end{aligned} \quad (23)$$

Here the following additional fields appear:

$$v^{(2)} = q * \Psi * \Psi * r, \quad (24)$$

$$q_{x_1} * \chi, \quad \tilde{\chi} * r_{x_1} \quad (25)$$

and we assume  $A^{(1)} = I$ . Eq.(21) holds for any pair  $(n, m)$ ,  $n \neq m$ . Putting  $m = 1$  in eqs.(21-23) and assuming  $C^{(1)} = 1$  we write them in the form

$$E_0^{(n1)} = v_{x_n} - v_{x_1} C^{(n)} + v A^{(n)} v - v v C^{(n)} - v^{(2)} [C^{(n)}, v], \quad (26)$$

$$F^{(n1)} = \tilde{\chi} * r_{x_1} A^{(n)} - \tilde{\chi} * r_{x_1} C^{(n)} + A^{(n)} q_{x_1} * \chi - q_{x_1} * \chi C^{(n)}. \quad (27)$$

The following combination of equations (21) is free of fields (25):

$$\left| \begin{array}{ccc} E_{\alpha\beta}^{(21)} & E_{\alpha\beta}^{(31)} & E_{\alpha\beta}^{(n1)} \\ A_{\beta}^{(2)} - C_{\beta}^{(2)} & A_{\beta}^{(3)} - C_{\beta}^{(3)} & A_{\beta}^{(n)} - C_{\beta}^{(n)} \\ A_{\alpha}^{(2)} - C_{\beta}^{(2)} & A_{\alpha}^{(3)} - C_{\beta}^{(3)} & A_{\alpha}^{(n)} - C_{\beta}^{(n)} \end{array} \right| = 0 \Leftrightarrow$$

$$\left| \begin{array}{ccc} (E_0^{(21)})_{\alpha\beta} & (E_0^{(31)})_{\alpha\beta} & (E_0^{(n1)})_{\alpha\beta} \\ A_{\beta}^{(2)} - C_{\beta}^{(2)} & A_{\beta}^{(3)} - C_{\beta}^{(3)} & A_{\beta}^{(n)} - C_{\beta}^{(n)} \\ A_{\alpha}^{(2)} - C_{\beta}^{(2)} & A_{\alpha}^{(3)} - C_{\beta}^{(3)} & A_{\alpha}^{(n)} - C_{\beta}^{(n)} \end{array} \right| = 0, \quad \alpha \neq \beta, \quad n = 4, 5, \dots \quad (28)$$

and

$$\left| \begin{array}{cc} E_{\alpha,\alpha}^{(21)} & E_{\alpha\alpha}^{(n1)} \\ A_{\alpha}^{(2)} - C_{\alpha}^{(2)} & A_{\alpha}^{(n)} - C_{\alpha}^{(n)} \end{array} \right| = 0 \Leftrightarrow$$

$$\left| \begin{array}{cc} (E_0^{(21)})_{\alpha\alpha} & (E_0^{(n1)})_{\alpha\alpha} \\ A_{\alpha}^{(2)} - C_{\alpha}^{(2)} & A_{\alpha}^{(n)} - C_{\alpha}^{(n)} \end{array} \right| = 0, \quad n = 3, 4, \dots \quad (29)$$

The hierarchy (28,29) involves two matrix fields  $v$  and  $v^{(2)}$ . In addition, eq.(28) is off-diagonal. Thus, the complete system of PDEs is represented, for instance by eq.(28) with  $n = 4, 5$  and by eqs.(29) with  $n = 3, 4$ . Therefore, the derived system of PDEs is five-dimensional.

### 2.2.1 Relations among non-diagonal elements of $v$ and $v^{(2)}$

Eq.(12) generates additional constraint for  $\Psi^{(2)}$  and  $\Psi$  as follows. Applying  $*\Psi$  to eq.(12) from the right we obtain

$$\Psi * Q^{(n)} * \Psi - Q^{(n)} * \Psi^{(2)} = \chi C^{(n)} \tilde{\chi}^{(2)}. \quad (30)$$

Using eq.(12) we can write it in the following form:

$$\Psi^{(2)} * Q^{(n)} - Q^{(n)} * \Psi^{(2)} = \chi^{(2)} C^{(n)} \tilde{\chi} + \chi C^{(n)} \tilde{\chi}^{(2)}. \quad (31)$$

Now we apply  $q*$  from the left and  $*r$  from the right to obtain

$$\tilde{E}^{(n)} := \tilde{\chi}^{(2)} * Q^{(n)} * r - q * Q^{(n)} * \chi^{(2)} - (v^{(2)} C^{(n)} v + v C^{(n)} v^{(2)}) = 0. \quad (32)$$

Owing to eqs.(13) and (14) the following combination of equations (32) is free of the terms with  $Q^{(n)}$ :

$$\sum_{perm(k,n,m)} (C^{(k)} \tilde{E}^{(n)} C^{(m)} - C^{(k)} \tilde{E}^{(m)} C^{(n)}), \quad (33)$$

which yields the following algebraic relation between  $v^{(2)}$  and  $v$ :

$$\sum_{perm(k,n,m)} \left( C^{(k)}(v^{(2)}C^{(n)}v + vC^{(n)}v^{(2)})C^{(m)} - C^{(k)}(v^{(2)}C^{(m)}v + vC^{(m)}v^{(2)})C^{(n)} \right) = 0. \quad (34)$$

Since  $C^{(1)} \equiv I$ , we can rewrite eq.(34) putting  $m = 1$  as follows:

$$\begin{aligned} & C^{(k)}v^{(2)}[C^{(n)}, v] + C^{(k)}v[C^{(n)}, v^{(2)}] - C^{(n)}v^{(2)}[C^{(k)}, v] - C^{(n)}v[C^{(k)}, v^{(2)}] + \\ & [v^{(2)}C^{(k)}, vC^{(n)}] + [vC^{(k)}, v^{(2)}C^{(n)}] = 0, \quad n \neq k. \end{aligned} \quad (35)$$

### 3 Solutions of system (28,29) with constraint (35)

In this section we find a particular form of functions  $Q$  associated with particular solutions of the system eq.(3,11,6,7,13-17). We take the following solution of system (6,7):

$$q(x; \mu) = \int d\Omega(k) e^{-k \sum_i A^{(i)} x_i} q_0(k, \mu), \quad r(x; \lambda) = \int d\Omega(k) r_0(\lambda, k) e^{k \sum_i A^{(i)} x_i}, \quad (36)$$

where  $d\Omega(k)$  is some measure on the complex plane of the parameter  $k$ . Solution of eq.(3) compatible with constraint (11) reads:

$$R(x; \lambda, \mu) = \int d\Omega(k) d\Omega(\tilde{k}) \frac{r_0(\lambda, k) e^{(k-\tilde{k}) \sum_i A^{(i)} x_i} q_0(\tilde{k}, \mu)}{k - \tilde{k}} + \sum_i x_i Q^{(i)}(\lambda, \mu) + I\delta(\lambda - \mu), \quad (37)$$

where  $A^{(1)}$  is the identity matrix.

Now we consider solution of eqs.(16) and (17). First, assuming invertibility of  $r_0$  and  $q_0$  we obtain from eq.(17):

$$Q^{(n)}(\lambda, \mu) = \int d\Omega(k) q_0^{-1}(\lambda, k) k C^{(n)} q_0(k, \mu). \quad (38)$$

Then eq.(16) requires the following relation between  $r_0$  and  $q_0$ :

$$r_0(\lambda, k) * q_0(k, \mu) = I\delta(\lambda - \mu) \quad (39)$$

so that eqs.(13-15) become identities.

#### 3.1 Degenerate functions $r_0(\lambda, k)$ and $q_0(k, \mu)$

Next, we consider the degenerate operators  $r_0$  and  $q_0$  with the purpose of construction the explicit solutions:

$$r_0(\lambda, k) = \sum_i r_{1i}(\lambda) r_{2i}(k), \quad q_0(k, \lambda) = \sum_i q_{1i}(k) q_{2i}(\lambda). \quad (40)$$

With  $r_0$  and  $q_0$  from (40), eqs. (16) and (17) can not be solved in general and, consequently, the functions  $v$  and  $v^{(2)}$  can not be constructed. This fact defers our equations from the classical soliton equations, where the arbitrary degenerate kernel of integral operator results in the

explicite formulas for the solutions of PDEs. But (16) and (17) can be solved if we take scalar  $r_{2i}$  and  $q_{1i}$  in the form of  $\delta$ -functions:

$$r_{2i}(k) = \delta(k - a_i), \quad q_{1i}(k) = \delta(k - b_i), \quad (41)$$

where  $a_i, b_i$  are scalar complex constants. Then equations (16-15) require the following structure for  $Q^{(n)}$ :

$$Q^{(n)}(\lambda, \mu) = \sum_i r_{1i}(\lambda) b_i C^{(n)} q_{2i}(\mu), \quad (42)$$

with

$$q_{2i} * r_{1j} = I \delta_{ij}. \quad (43)$$

### 3.1.1 Explicite expression for field $v$

For the degenerate functions (40,41), we can write the kernel  $R$  (37) in the form:

$$R(\lambda, \mu) = \sum_{i,j} r_{1i}(\lambda) R_{ij}^{(0)} q_{2j}(\mu) + I \delta(\lambda - \mu), \quad (44)$$

$$R_{ij}^{(0)} = \frac{e^{(a_i - b_i) \sum_m A^{(m)} x_m}}{a_i - b_j} + \sum_n x_n C^{(n)} b_j \delta_{ij},$$

where  $\delta_{ij}$  is the Kronecker symbol. Substituting this  $R$  into the evident equation  $(R * \Psi)(\lambda, \mu) = I \delta(\lambda - \mu)$  we obtain

$$\Psi(x; \lambda, \mu) = I \delta(\lambda - \mu) - \sum_{i,j} r_{1i}(\lambda) R_{ij}^{(0)}(x) \Psi_j(x; \mu) \quad (45)$$

where  $\Psi_j = q_{2j} * \Psi$ . Applying  $q_{2k} *$  to eq.(45) from the left we obtain the linear algebraic equation for  $\Psi_j$ :

$$\Psi_k(x; \mu) = q_{2k}(\mu) - \sum_j R_{kj}^{(0)}(x) \Psi_j(x; \mu), \quad k = 1, 2, \dots, \quad (46)$$

or

$$\sum_j \hat{R}_{kj} \Psi_j(x; \mu) = q_{2k}(\mu), \quad \hat{R}_{kj} = I \delta_{kj} + R_{kj}^{(0)}. \quad (47)$$

which yields

$$\Psi_j(x; \mu) = \sum_k (\hat{R}^{-1}(x))_{jk} q_{2k}(\mu), \quad (48)$$

where  $\hat{R}$  has the following block structure:  $\hat{R} = \{\hat{R}_{kj}^{(0)}\}$ . Substituting (48) into eq.(45) we obtain

$$\Psi(x; \lambda, \mu) = I \delta(\lambda - \mu) - \sum_{i,j,n} r_{1i}(\lambda) R_{ij}^{(0)}(x) (\hat{R}^{-1}(x))_{jn} q_{2n}(\mu). \quad (49)$$

Now, applying  $q *$  and  $*r$  to (46) we obtain expression for  $v$ :

$$v = \sum_i e^{(a_i - b_i) \sum_m A^{(m)} x_m} + \sum_{i,j,n} e^{-b_i \sum_m A^{(m)} x_m} R_{ij}^{(0)}(x) (\hat{R}^{-1}(x))_{jn} e^{a_n \sum_l A^{(l)} x_l}. \quad (50)$$

Solution of form (50) can be referred to the solitary-wave solutions. Expression for  $v^{(2)}$  can be obtain in a similar way using its definition (24). We do not represent it here.

## 4 Conclusion

In this paper we propose a modification of the dressing method based on the Fredholm-type integral equation with the kernel of special type which, along with the Cauchy term exponentially depending on  $x$ , involves a term linear in  $x$ . This term allows us to increase the dimensionality of the associated nonlinear equations. We derive the five-order system of the first order PDEs (28,29) which is a multidimensional extension of the classical (2+1)-dimensional ISTM-integrable  $N$ -wave equation. This equation is supplemented by the algebraic constraint (35) We show that these equation posses a rich family of solutions including the explicite solutions which can be produced by the degenerate kernel of integral operator. Further increase of dimensionality is possible and will be discussed later.

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